Heat Transfer Characteristics of Fluids Moving in a Taylor System of Vortices

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Part I. $N_{Pr} = 1$

G. I. Taylor (*7*) investigated both experimentally and theoretically the instability of viscous isothermal flow between rotating cylinders and found that when the speed exceeded a critical value for a given geometry of apparatus, toroidal vortices were generated in the annulus. The critical parameter is called the *critical Taylor number*, and for the case of an outer stationary cylinder with a very narrow gap width (that is $d << r_m$) it may be written as*

$$N_{Ta_c}{}^* = \frac{\Omega_1{}^2 r_m d^3}{\nu^2} = 1{,}708$$

The purpose of the present work is to study the heat transfer characteristics of fluids in systems of Couette motion with and without vortices. A number of authors have already dealt with this problem.

Bjorklund and Kays (*1*) gave comprehensive heat transfer data for air and found that for the inner cylinder only rotating their data were correlated by the equation

$$\frac{N_{Nu}}{N_{Nu_k}} = 0.175\, N_{Ta}{}^{1/2} \quad 90 < N_{Ta} < 2{,}000$$

Becker and Kaye (*8*) and Tachibana, Fukui, and Mitsumura (*6*) did further extensive experimental work on heat transfer for air which showed excellent agreement with that of Bjorklund and Kays. Similarly work by Haas and Nissan (*3, 4*) dealt with liquids, presenting heat transfer characteristics of water and of several glycerol-water solutions of different concentration. However, the results of these works are experimental in nature.

In the present work a theoretical prediction of heat transfer characteristics for fluids with $N_{Pr} = 1$ is given in Part 1. In Part II an empirical relation for the heat transfer characteristics for fluids with $N_{Pr} \neq 1$ is obtained experimentally. In addition a comprehensive study of the temperature profiles in the Taylor system is made.

DESCRIPTION OF APPARATUS AND PROCEDURE

A description of the apparatus used in Parts I and II is given below.

The test equipment consisted of an inner rotating cylinder and an outer stationary cylinder coaxial with it. The inner rotating cylinder was heated by an interior electric heater; the outer stationary cylinder was surrounded by a water jacket for dissipating the heat transmitted through it. Figure 1 presents the outline of the test equipment.

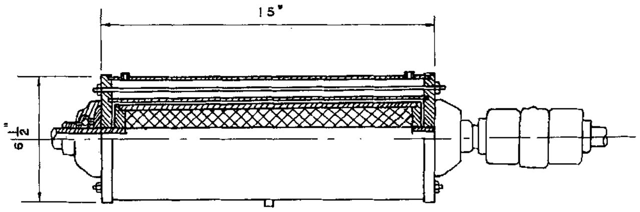

Fig. 1. Schematic diagram of apparatus.

Inner Cylinder

The inner cylinder was turned from stainless steel pipe and was 15 in. long, with an O.D. of 3.470 ± 0.001 in. This cylinder contained eleven iron constantan thermocouples embedded at its external surface. A slip ring assembly was used to transmit the signal. The inner cylinder was heated by an aluminum conduction cylinder which had a Chromalox cartridge heater inserted inside it. Heater power was supplied through a Variac to slip rings on the cylinder. Rotation was provided by a v-belt drive from a ¾ h.p. hydraulic transmission unit with variable speed from 0 to 2,000 rev./min. in either direction. The end plates at both ends were made of Teflon.

Outer Cylinder (for heat transfer measurements)

An outer stainless steel cylinder with an I.D. of 3.753 ± 0.001 in. was used. The cylinder contained eleven iron constantan thermocouples placed at its internal surface. The outside cylinder assembly was held concentric with the inner cylinder and the water jacket by means of bolts holding it to the end plates.

Temperature Probe

Equipment was designed for the special purpose of making temperature traverses in both radial and axial directions. A diagram of the equipment is shown in Figure 2. This equip-

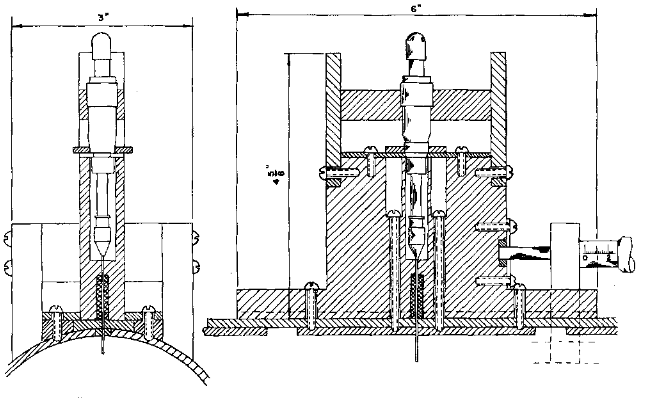

Fig. 2. Temperature probe.

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* Many workers prefer to use the square root of N_{Ta}* as defined above and write the Taylor number as

$$\frac{\Omega_1 r_m d}{\nu}\sqrt{\frac{d}{r_m}} \qquad \left(\text{for } \frac{d}{r_m} << 1,\ r_1 \equiv r_m\right)$$

The authors shall denote this number as N_{Ta}.

ment was mounted on a cast acrylic resin tube, with all surfaces polished, and had an I.D. of 4.225 ± 0.005 in., thus the gap width was 0.377 ± 0.006 in.

A 36 B and S gauge iron constantan thermocouple was used to measure the temperature distribution within the gap. The Teflon coated wire was placed in a No. 22 hypodermic needle with the thermocouple bead extending from the tip of the needle. The needle was supported in a movable carriage. Two micrometer screws fastened to this equipment were used to drive the needle. The thermocouple lead wires were connected directly to a potentiometer to measure the temperature.

Procedure

The heat transfer data were taken in the following manner. After the annulus was filled with the fluid under study, the heater was energized. When the system reached a steady state, the temperature of each cylinder was recorded. For each subsequent reading the inner cylinder was rotated at the desired speed until steady state had again been reached. The temperature of each cylinder was then recorded.

The temperature profiles were taken by moving the recording thermocouple radially inwards in small increments for various settings of cylinder speed and various fluids. The temperature at each radial position was recorded.

THEORETICAL RELATIONS AND RESULTS ($N_{Pr} = 1$)

The Prandtl number of most gases is close to unity. Under small temperature differences the variations in viscosity, thermal diffusivity, and density will be relatively small.

Thus, assuming constant properties and neglecting viscous dissipation, one can write the basic energy equation as:

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right) T = -\overline{V}\cdot\nabla T \qquad (1)$$

It is known from the experimental results of this work and of others that when the rotational speed of the cylinder is above a critical value (or more precisely when the Taylor number exceeds a critical value) the mode of heat transfer between the cylinders changes from molecular conduction to convection, and the steady laminar flow which appears takes the form of cellular, toroidal vortices spaced regularly along the axis of the cylinder.

If rotational symmetry is assumed in the annulus $\left(\frac{\partial}{\partial\theta} = 0\right)$, then the energy equation is

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = K\nabla^2 T$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad (2)$$

Now consider the complete energy equation, Equation (1), in these coordinates for the quantities u, v, w, T (the three velocity components and the temperature, respectively). To clarify the role of the nonlinear term $V \cdot \nabla T$ divide each quantity $q(r, z, t)$ into a mean value part and a disturbance q' as

$$q(r, z, t) = \overline{q}(r,t) + q'(r, z, t) \qquad (3)$$

or write them specifically as

$$u = 0 + u'$$
$$v = \overline{v} + v'$$
$$w = 0 + w'$$
$$T = \overline{T} + t'$$

Undisturbed v and T are

$$v = Ar + \frac{B}{r}$$
$$T = C\ln r + D$$

Since the flow which results from the instability is periodic with respect to z, it is convenient to average with respect to z. In order to allow for the distortion of the disturbance by nonlinearity one writes

$$q' = q_1(r,t)\,e^{i\alpha z} + q_2(r,t)\,e^{2i\alpha z} + \ldots + \tilde{q}_1(r,t)\,e^{-i\alpha z} + \tilde{q}_2(r,t)\,e^{-2i\alpha z} + \ldots \qquad (5)$$

where the symbol ~ denotes a complex conjugate.

If the above series is substituted into Equation (2) and the Fourier components are separated, there results a set of equations of all the functions involved in Equation (5). For the mean motion

$$\frac{\partial\overline{T}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{t'w'}) = K\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\overline{T} \qquad (6)$$

The dependence of $\overline{T}$ on t is retained because if the disturbance is growing or decaying, the mean motion undergoes distortion in order to maintain the energy balance. If r_1 and r_2 are the radii of inner and outer cylinders and T_1 and T_2 their temperatures, the boundary conditions are

$$\overline{T} = T_1 \quad \text{at } r = r_1$$
$$\overline{T} = T_2 \quad \text{at } r = r_2 \qquad (7)$$

In the state of equilibrium $\partial\overline{T}/\partial t = 0$, and Equation (6) can be integrated to give:

$$\overline{T} = C'\ln r + D' + \frac{1}{K}\int_{r_1}^{r}\overline{t'u'}\,dr \qquad (8)$$

where

$$C' = \frac{T_1 - T_2}{\ln r_1/r_2} + \frac{1}{\ln r_1/r_2}\frac{1}{K}\int_{r_1}^{r_2}\overline{t'u'}\,dr \qquad (9)$$

$$D' = T_1 - \frac{(T_1 - T_2)\ln r_1}{\ln r_1/r_2} + \frac{\ln r_1}{\ln r_1/r_2}\frac{1}{K}\int_{r_1}^{r_2}\overline{t'u'}\,dr \qquad (10)$$

In addition to the mean energy equation it is possible to write down an infinite set of differential equations for the harmonic components of the disturbance. These components are all mutually dependent because of the nonlinearity of the system. For purposes of approximation the assumption is made that the first harmonic component of the disturbances (u_1, v_1, w_1, t_1) will be similar in shape to that given by linear theory but multiplied by an amplitude factor.

It is necessary then to linearize the energy equation and to find the shape function for t_1. In the usual manner let

$$\left.\begin{aligned} u &= o + u_1(r)\cos\alpha z\,e^{\beta t} \\ v &= \overline{v} + v_1(r)\cos\alpha z\,e^{\beta t} \\ w &= o + w_1(r)\sin\alpha z\,e^{\beta t} \\ T &= \overline{T} + t_1(r)\cos\alpha z\,e^{\beta t} \end{aligned}\right\} \qquad (11)$$

It is now convenient to introduce the following units for velocity, temperature, and linear dimension:

$$\left.\begin{aligned} &\text{Velocity: } \Omega_1 r_1 \\ &\text{Temperature: } T_1 - T_2 = \Delta T \\ &\text{Distance: } d \end{aligned}\right\} \qquad (12)$$

It is convenient also to introduce the following dimensionless quantities:

$$
\left.
\begin{aligned}
&\lambda = \alpha\, d, \quad \sigma = \beta\, d^2/v, \\
&\tau = \frac{r - r_m}{d} \text{ where } r_m = (r_1 + r_2)/2 \\
&N_{Pe} = r_1\, \Omega_1\, d/k = \text{Peclet number} \\
&N_{Pr} = \frac{\nu}{K} = \text{Prandtl number} \\
&N_{Ta^*} = \left(\frac{\Omega_1\, r_m\, d}{\nu} \sqrt{\frac{d}{r_m}} \right)^2 = \text{Taylor number} \\
&N_{Re} = \frac{r_1\, \Omega_1\, d}{v} = \text{Reynolds number}
\end{aligned}
\right\} \quad (13)
$$

These units and quantities will be adapted in the following linear analysis from Equation (14) to Equation (19). Substitute Equation (11) into Equation (2) and neglect all terms containing squares or products of disturbance or their derivatives. Let the annular gap be very small compared with the radii of cylinder, and restrict the analysis to the case of $\Omega_2 = o$ (that is outer cylinder is stationary). Then the linear energy equation, with respect to the disturbance, can be written as

$$(D^2 - \lambda^2 - N_{Pr}\, \sigma)\; t_1 = N_{Pe} \frac{d\overline{T}}{d\tau} u_1 \qquad (14)$$

where $D = d/d\tau$ with boundary conditions $t_1 = o$ at $\tau = \pm\, ½$.

To the same approximation ($d << r_m$) the mean temperature distribution at the critical state is $\overline{T} = T_1 + ½\,(1 - 2\tau)$, and therefore $d\overline{T}/d\tau = -1$. Equation (14) becomes

$$(D^2 - \lambda^2 - N_{Pr}\, \sigma)\; t_1 = -N_{Pe}\, u_1 \qquad (15)$$

The linear stability problem under assumptions $d/r_m \to o$ and $\Omega = (\Omega_1 + \Omega_2)/2$ reduces to [see Chandrasekhar (2)]

$$(D^2 - \lambda^2)\;(D^2 - \lambda^2 - \sigma)\; u_1 = \lambda^2\, Ta\, v_1 \qquad (16)$$

$$(D^2 - \lambda^2 - \sigma)\; v_1 = -u_1 \qquad (17)$$

with boundary conditions $v_1 = D^2 v_1 = D(D^2 - \lambda^2 - \sigma)\, v_1 = o$ at $\tau = \pm\, ½$.

In comparing Equation (17) with Equation (15) it is seen that when $N_{Pr} = 1$ (for most gases the Prandtl number is close to unity), t_1 and v_1 are determined by the same type of differential equation and also with similar boundary conditions. It follows that there is an analogy between t_1 and v_1:

$$t_1 = N_{Pe}\, v_1 \qquad (18)$$

Since for $N_{Pr} = 1$, $N_{Pe} = N_{Re}$ Equation (18) can also be written

$$t_1 = N_{Re}\, v_1 \qquad (19)$$

J. E. Stuart (5) investigated the nonlinear mechanics of flow between rotating cylinders under supercritical conditions on the basis of a finite amplitude theory. He found that mean flow was given by

$$\overline{V} = 1/2\, r_1 \Omega_1\, (1 - 2\tau) - (a_e^2 r_1^3 \Omega_1^3 d^3/\nu^2)\, Q(\tau) \qquad (20)$$

$$Q(\tau) = -2 \int_{-1/2}^{\tau} Z\, d\tau + 2\tau \int_{-1/2}^{1/2} Z\, d\tau + \int_{-1/2}^{1/2} Z\, d\tau \qquad (21)$$

$$
\left.
\begin{aligned}
&Z = -S\; (D^2 - \lambda^2) S \\
&u_1 = -a_e\, r_1\, \Omega_1\; (D^2 - \lambda^2) S \\
&v_1 = (a_e\, r_1^2\, \Omega_1^2\, d/\nu) S
\end{aligned}
\right\} \quad (22)
$$

where $S(\tau)$ is the shape function. $S(\tau)$ will be assumed to be the function appropriate to $\sigma = o$, $\lambda = \lambda_c = \pi$, $N_{Ta}{}^* = 1{,}708$. This is the case of disturbance at critical Taylor number for which one has, from the variational condition given by Chandrasehar (2)

$$S = \frac{-1}{8\pi^2} + \frac{1}{\pi^4} + \frac{\tau^2}{2\pi^2} + \frac{\cos \pi\tau}{4\pi^3} - \frac{0.02686}{2\pi^3} \left(1 + \frac{1}{5} \cos 2\pi\tau \right) - \frac{1}{4}\left(1 + \frac{2\pi}{5} \right) \frac{\cosh \pi\tau}{\cosh \frac{1}{2}\pi} \qquad (23)$$

and a_e the equilibrium amplitude which has been calculated by Stuart (5). Thus

$$a_e^2 = \frac{5.425 \times 10^4}{N_{Re}^2} \left(1 - \frac{1{,}708}{N_{Ta^*}} \right) \qquad (24)$$

To simplify Equation (8) let ($d/r_m \to o$) and adopt Stuart's symbols. The mean temperature distribution is given by

$$\overline{T} = T_1 + \frac{1}{2} \Delta T (1 - 2\tau) - a_e^2\; N_{Re}^2\, \Delta T\, Q(\tau) \qquad (25)$$

$$t_1 = (a_e\; N_{Re}\; \Delta\; T)\;\; S \qquad (26)$$

Equation (25) gives the temperature distribution. The rate of heat transfer can be simply found by applying Fourier's law of heat conduction; that is

$$q = -k \frac{d\overline{T}}{dn} = -K \frac{d\overline{T}}{dr}\bigg|_{r_1} = -\frac{k}{d} \frac{d\overline{T}}{d\tau}\bigg|_{\tau = ½} \qquad (27)$$

$$\frac{d\overline{T}}{d\tau}\bigg|_{½} = -\Delta T \left[1 + 5.245 \times 10^4 \frac{dQ\;(\tau)}{d\tau} \left(1 - \frac{1{,}708}{N_{Ta^*}} \right) \right] = -\Delta T \left[1 + 1.4472 \left(1 - \frac{1{,}708}{N_{Ta^*}} \right) \right] \qquad (28)$$

and

$$q = \frac{k}{d} \Delta T \left[1 + 1.4472 \left(1 - \frac{1{,}708}{N_{Ta^*}} \right) \right] \qquad (29)$$

The heat transfer coefficient h is defined as

$$h = \frac{Q}{A_i(T_1 - T_2)} = \frac{q}{(T_1 - T_2)} \qquad (30)$$

where Q is the total heat transfer rate, and A_i is the inner cylinder surface area. The Nusselt number of an annulus is defined as $\frac{h(D_2 - D_1)}{k} = \frac{2hd}{k}$. For streamline flow (conduction heat transfer) the Nusselt number is dependent only on geometry, that is

$$N_{Nuk} = \frac{2 \dfrac{d}{r_1}}{\ln \left(\dfrac{d}{r_1} + 1 \right)} \qquad (31)$$

For $d/r_1 \to o$ this may be reduced to $N_{Nuk} \cong 2$.

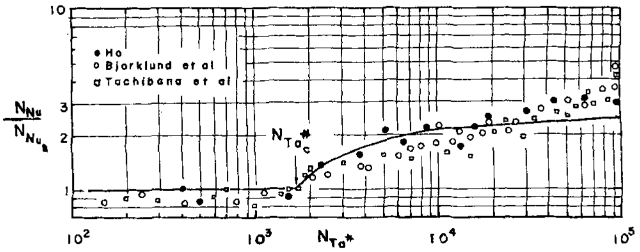

Fig. 3. Comparison of theory with experiment.

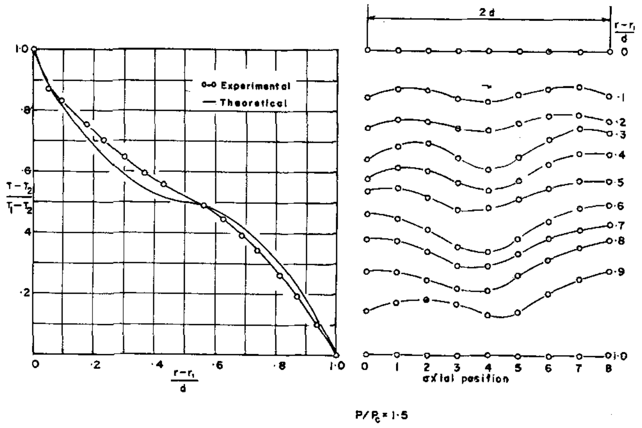

Fig. 4a. Radial and axial temperature profiles.

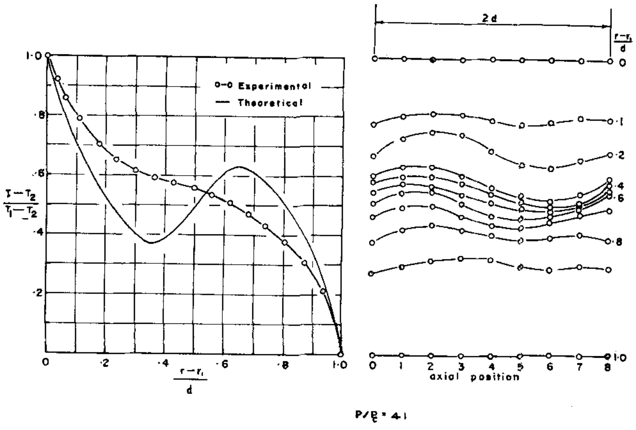

Fig. 4b. Radial and axial temperature profiles.

One has the rate of heat transfer from Equation (29). The Nusselt number is given by

$$N_{Nu} = \frac{2hd}{k} = 2\left[1 + 1.4472\left(1 - \frac{1{,}708}{N_{Ta^*}}\right)\right] \quad (32)$$

and

$$\frac{N_{Nu}}{N_{Nuk}} = 1 + 1.4472\left(1 - \frac{1{,}708}{N_{Ta^*}}\right) \quad (33)$$

Since the Prandtl number for air is close to unity (that is $N_{Pr} = 0.7$), the above theoretical analysis may be applied. Figure 3 shows the comparison of theoretical analysis to the experimental results of air, and it can be seen that the agreement is satisfactory.

DISCUSSION AND CONCLUSION

Figure 3 shows that the theoretical analysis for air gives a reasonably correct result for the heat transfer for a short range of Taylor numbers above the critical value. It is believed that the gradual divergence between theoretical and experimental values as the Taylor number increases is due to the increasing divergence from reality of the simplifying assumption made in the derivation; that is velocity and temperature distributions become increasingly different in fact from those assumed in the theory as the Taylor number increases.

Thus Figure 4*a* shows the temperature profiles as measured and as predicted theoretically for a relatively low Taylor number. The differences observed are relatively small and are probably due as much to the difficulties of accurate temperature measurements in the fluid as to theoretical errors. On the other hand Figure 4*b* shows that the measured and theoretically predicted profiles are completely different for a high Taylor number. The reversal of sign of the temperature gradient in the central region of the theoretically predicted profile is clearly not realistic, and the theory fails.

Presented along with the radial temperature profiles in Figure 4 are axial temperature profiles. The periodic variation of temperature in an axial direction can easily be seen; this would indicate that the temperature within the vortex is nearly constant axially but with some sinusoidal disturbance.

In Part II the analysis for substances with $N_{Pr} = 1$ will be extended to include a higher range of Taylor numbers. In addition the heat transfer characteristics of fluids with Prandtl numbers different from 1 will also be investigated.

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Part II. $N_{Pr} \neq 1$

In Part I a review of the literature was made. Although several authors (*1, 2, 12*) have studied the heat transfer characteristics of air in the Taylor system, there has been little work on fluids in general. In an attempt to generalize the heat transfer relations for fluids of various properties Haas and Nissan (*7*) presented heat transfer data for water and several glycerol-water solutions of different concentration. They found for the fluids they studied that

$$\frac{N_{Nu}}{N_{Nuk}} = \left(\frac{P}{P_c}\right)^m$$

where P is a form of Taylor number, and m depends on the fluid properties.

Ho, Nardacci, and Nissan (*10*) obtained an analytical relation which predicted the heat transfer characteristics of the Taylor vortex system for a small range of Taylor numbers above the critical value. However the theoretical analysis is only valid for air or for those fluids with a Prandtl number close to unity; it has the further restriction of narrow gap geometry and of ΔT being small.

The purpose of the present work is to extend the previous heat transfer results and to obtain a generalized empirical correlation for fluids of widely differing properties. In addition a more complete study of the temperature distribution in the annulus for various fluids is presented.